

Jeśmanowicz' conjecture revisited,II

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Abstract. Let a, b, c be relatively prime positive integers such that $a^2 + b^2 = c^2$. In 1956, Jeśmanowicz conjectured that for any positive integer n , the only solution of $(an)^x + (bn)^y = (cn)^z$ in positive integers is $(x, y, z) = (2, 2, 2)$. Let $k \geq 1$ be an integer and $F_k = 2^{2^k} + 1$ be a Fermat number. In this paper, we show that Jeśmanowicz' conjecture is true for Pythagorean triples $(a, b, c) = (F_k - 2, 2^{2^{k-1}+1}, F_k)$.

Keywords: Jeśmanowicz' conjecture; Diophantine equation

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1 Introduction

Let a, b, c be relatively prime positive integers such that $a^2 + b^2 = c^2$ with $2 \mid b$. Clearly, the Diophantine equation

$$(na)^x + (nb)^y = (nc)^z \quad (1.1)$$

has the solution $(x, y, z) = (2, 2, 2)$. In 1956, Sierpiński [7] showed there is no other solution when $n = 1$ and $(a, b, c) = (3, 4, 5)$, and Jeśmanowicz [2] proved that when $n = 1$ and $(a, b, c) = (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)$, the Eq.(1.1) has only the solution $(x, y, z) = (2, 2, 2)$. Moreover, he conjectured that for any positive integer n , the Eq.(1.1) has no solution other than $(x, y, z) = (2, 2, 2)$. Let $k \geq 1$ be an integer and $F_k = 2^{2^k} + 1$

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be a Fermat number. Recently, the first author of this paper and Yang [8] proved that if $1 \leq k \leq 4$, then the Diophantine equation

$$((F_k - 2)n)^x + (2^{2^{k-1}+1}n)^y = (F_k n)^z \quad (1.2)$$

has no solution other than $(x, y, z) = (2, 2, 2)$. For related problems, see ([1], [5], [6]).

In this paper, we obtain the following result.

Theorem 1. *Let k be a positive integer and $F_k = 2^{2^k} + 1$ be a Fermat number. Then the Eq.(1.2) has only the solution $(x, y, z) = (2, 2, 2)$.*

Throughout this paper, let m be a positive integer and a be any integer relatively prime to m . If h is the least positive integer such that $a^h \equiv 1 \pmod{m}$, then h is called the order of a modulo m , denoted by $\text{ord}_m(a)$.

2 Lemmas

Lemma 1. ([4]) *The Diophantine equation $(4m^2 - 1)^x + (4m)^y = (4m^2 + 1)^z$ has only the solution $(x, y, z) = (2, 2, 2)$.*

Lemma 2. (See [1, Lemma 2]) *If $z \geq \max\{x, y\}$, then the Diophantine equation $a^x + b^y = c^z$, where a, b and c are any positive integers (not necessarily relative prime) such that $a^2 + b^2 = c^2$, has no solution other than $(x, y, z) = (2, 2, 2)$.*

Lemma 3. (See [3, Corollary 1]) *If the Diophantine equation $(na)^x + (nb)^y = (nc)^z$ (with $a^2 + b^2 = c^2$) has a solution $(x, y, z) \neq (2, 2, 2)$, then x, y, z are distinct.*

Lemma 4. *Let a, b, c be relatively prime positive integers with $2 \mid b$ and $c \leq 3a$. If the Diophantine equation $a^x + b^y = c^z$ has only the solution $(x, y, z) = (2, 2, 2)$, then the Eq.(1.1) has no solution (x, y, z) satisfying $z < \min\{x, y\}$.*

Proof. We may suppose that $n \geq 2$ and the Eq.(1.1) has a solution (x, y, z) satisfying $z < \min\{x, y\}$. By Lemma 3, it is sufficient to consider the following two cases:

Case 1. $x < y$. Then we have

$$n^{x-z}(a^x + b^y n^{y-x}) = c^z. \quad (2.1)$$

It is clear from (2.1) that $\gcd(n, c) > 1$. Let $c = \prod_{i=1}^t p_i^{\alpha_i}$ be the stand factorization of c and write $n = \prod_{\nu=1}^s p_{i_\nu}^{\beta_{i_\nu}}$, where $\beta_{i_\nu} \geq 1$, $\{i_1, \dots, i_s\} \subseteq \{1, \dots, t\}$. By (2.1), we have

$$\prod_{\nu=1}^s p_{i_\nu}^{\beta_{i_\nu}(x-z)} \left(a^x + b^y \prod_{\nu=1}^s p_{i_\nu}^{\beta_{i_\nu}(y-x)} \right) = \prod_{i=1}^t p_i^{\alpha_i z}. \quad (2.2)$$

Noting that

$$\gcd \left(a^x + b^y \prod_{\nu=1}^s p_{i_\nu}^{\beta_{i_\nu}(y-x)}, \prod_{\nu=1}^s p_{i_\nu} \right) = 1,$$

we know that

$$a^x + b^y \prod_{\nu=1}^s p_{i_\nu}^{\beta_{i_\nu}(y-x)} = \prod_{i \in T} p_i^{\alpha_i z}, \quad (2.3)$$

where $T = \{1, 2, \dots, t\} \setminus \{i_1, \dots, i_s\}$.

Since $2 \mid b$, we have $2 \nmid c$, thus

$$\prod_{i \in T} p_i^{\alpha_i z} \leq \left(\frac{c}{3} \right)^z < \left(\frac{c}{3} \right)^x \leq a^x,$$

which contradicts with (2.3).

Case 2. $x > y$. Then we have

$$n^{y-z} \left(b^y + a^x n^{x-y} \right) = c^z. \quad (2.4)$$

The remainder of the proof is similar to that of the proof of Case 1. We omit it here.

This completes the proof of Lemma 4. \square

Lemma 5. *Let k be a positive integer and $F_k = 2^{2^k} + 1$ be a Fermat number. If (x, y, z) is a solution of the Eq.(1.2) with $(x, y, z) \neq (2, 2, 2)$, then $x < z < y$.*

Proof. By Lemmas 2-4, it is sufficient to prove that the Eq.(1.2) has no solution (x, y, z) satisfying $y < z < x$. By Lemma 1, we may suppose that $n \geq 2$ and the Eq.(1.2) has a solution (x, y, z) with $y < z < x$. Then we have

$$2^{(2^{k-1}+1)y} = n^{z-y} \left(F_k^z - (F_k - 2)^x n^{x-z} \right). \quad (2.5)$$

By (2.5) we may write $n = 2^r$ with $r \geq 1$. Noting that

$$\gcd \left(F_k^z - (F_k - 2)^x 2^{r(x-z)}, 2 \right) = 1,$$

we have

$$F_k^z - (F_k - 2)^x 2^{r(x-z)} = 1. \quad (2.6)$$

Since $k \geq 1$, by (2.6) we have $F_k^z \equiv 1 \pmod{3}$, $z \equiv 0 \pmod{2}$. Write $z = 2z_1$, we have

$$\left(\prod_{i=0}^{k-1} F_i \right)^x 2^{r(x-z)} = (F_k^{z_1} - 1)(F_k^{z_1} + 1). \quad (2.7)$$

Let $F_{k-1} = \prod_{i=1}^t p_i^{\alpha_i}$ be the standard factorization of F_{k-1} with $p_1 < \dots < p_t$. Then

$$\text{ord}_{p_i}(2) = 2^k, \quad i = 1, \dots, t. \quad (2.8)$$

Noting that $\gcd(F_k^{z_1} - 1, F_k^{z_1} + 1) = 2$, we know that p_t divide only one of $F_k^{z_1} - 1$ and $F_k^{z_1} + 1$.

Case 1. $p_t \mid F_k^{z_1} - 1$. Then $F_k^{z_1} - 1 \equiv 2^{z_1} - 1 \equiv 0 \pmod{p_t}$. Noting that $\text{ord}_{p_t}(2) = 2^k$, we have $z_1 \equiv 0 \pmod{2^k}$. By (2.8) we have

$$F_k^{z_1} - 1 \equiv 2^{z_1} - 1 \equiv 0 \pmod{p_i}, \quad i = 1, \dots, t.$$

Since $\gcd(F_k^{z_1} - 1, F_k^{z_1} + 1) = 2$, by (2.7) we have

$$F_k^{z_1} - 1 \equiv 2^{z_1} - 1 \equiv 0 \pmod{p_i^{\alpha_i x}}, \quad i = 1, \dots, t.$$

Hence $F_{k-1}^x \mid F_k^{z_1} - 1$.

Case 2. $p_t \mid F_k^{z_1} + 1$. Then $F_k^{z_1} + 1 \equiv 2^{z_1} + 1 \equiv 0 \pmod{p_t}$. Noting that $\text{ord}_{p_t}(2) = 2^k$, we have $2^{k-1} \mid z_1$, but $2^k \nmid z_1$. By (2.8) we have

$$2^{2z_1} - 1 = (2^{z_1} + 1)(2^{z_1} - 1) \equiv 0 \pmod{p_i}, \quad i = 1, \dots, t.$$

Thus

$$F_k^{z_1} + 1 \equiv 2^{z_1} + 1 \equiv 0 \pmod{p_i}, \quad i = 1, \dots, t.$$

Since $\gcd(F_k^{z_1} - 1, F_k^{z_1} + 1) = 2$, by (2.7) we have

$$F_k^{z_1} + 1 \equiv 2^{z_1} + 1 \equiv 0 \pmod{p_i^{\alpha_i x}}, \quad i = 1, \dots, t.$$

Hence $F_{k-1}^x \mid F_k^{z_1} + 1$.

However,

$$F_{k-1}^x = \left(2^{2^{k-1}} + 1 \right)^x > \left(2^{2^{k-1}} + 1 \right)^{2z_1} > F_k^{z_1} + 1,$$

which is impossible.

This completes the proof of Lemma 5. □

3 Proof of Theorem 1

By Lemma 1 and Lemma 5, we may suppose that $n \geq 2$ and the Eq.(1.2) has a solution (x, y, z) with $x < z < y$. Then

$$\left(\prod_{i=0}^{k-1} F_i \right)^x = n^{z-x} \left(F_k^z - 2^{(2^{k-1}+1)y} n^{y-z} \right). \quad (3.1)$$

It is clear from (3.1) that

$$\gcd \left(n, \prod_{i=0}^{k-1} F_i \right) > 1.$$

Let $\prod_{i=0}^{k-1} F_i = \prod_{i=1}^t p_i^{\alpha_i}$ be the standard factorization of $\prod_{i=0}^{k-1} F_i$ and write $n = \prod_{\nu=1}^s p_{i_\nu}^{\beta_{i_\nu}}$, where $\beta_{i_\nu} \geq 1$, $\{i_1, \dots, i_s\} \subseteq \{1, \dots, t\}$. Let

$$P(k, n) = \prod_{i \in T} p_i^{\alpha_i},$$

where $T = \{1, 2, \dots, t\} \setminus \{i_1, \dots, i_s\}$. By (3.1), we have

$$P(k, n)^x = F_k^z - 2^{(2^{k-1}+1)y} \prod_{\nu=1}^s p_{i_\nu}^{\beta_{i_\nu}(y-z)}. \quad (3.2)$$

Since $y \geq 2$, it follows that

$$P(k, n)^x \equiv 1 \pmod{2^{2^k}}. \quad (3.3)$$

If $3 \mid P(k, n)$, then $P(k, n) \equiv -1 \pmod{4}$. This implies that x is even. If $3 \nmid P(k, n)$, then $P(k, n) \equiv 1 \pmod{4}$. Let $P(k, n) = 1 + 2^v W$, $2 \nmid W$. Then $v \geq 2$. Suppose that x is odd, then

$$P(k, n)^x = 1 + 2^v W', \quad 2 \nmid W'.$$

Thus $v \geq 2^k$ and $P(k, n) \geq F_k$, a contradiction with

$$P(k, n) < \prod_{i=0}^{k-1} F_i = F_k - 2.$$

Therefore, x is even and $P(k, n) \equiv -1 \pmod{4}$.

Let $P(k, n) = 2^d M - 1$, $2 \nmid M$ and let $x = 2^u N$, $2 \nmid N$. Then $d \geq 2$ and $u \geq 1$. Thus

$$P(k, n)^x = 1 + 2^{u+d} V, \quad 2 \nmid V.$$

By (3.3) we have $u + d \geq 2^k$.

Choose a $\nu \in \{1, \dots, s\}$, let $p_{i_\nu} = 2^r t + 1$ with $r \geq 1$, $2 \nmid t$. Then

$$2^{d+r-1} < (2^d M - 1)(2^r t + 1) = P(k, n) \cdot p_{i_\nu} \leq \prod_{i=0}^{k-1} F_i = 2^{2^k} - 1.$$

Thus $d + r \leq 2^k$. Hence $u \geq r$. By (3.2) we have

$$P(k, n)^x \equiv 2^z \pmod{p_{i_\nu}}. \quad (3.4)$$

Noting that $p_{i_\nu} - 1 \mid 2^u t$, we have

$$2^{tz} \equiv P(k, n)^{2^u t N} \equiv 1 \pmod{p_{i_\nu}}, \quad (3.5)$$

Since $\text{ord}_{p_{i_\nu}}(2)$ is even and $2 \nmid t$, we have $z \equiv 0 \pmod{2}$.

Write $z = 2z_1$, $x = 2x_1$. By (3.2), we have

$$2^{(2^{k-1}+1)y} \prod_{\nu=1}^s p_{i_\nu}^{\beta_{i_\nu}(y-z)} = \left(F_k^{z_1} - P(k, n)^{x_1}\right) \left(F_k^{z_1} + P(k, n)^{x_1}\right). \quad (3.6)$$

Noting that

$$\gcd\left(F_k^{z_1} - P(k, n)^{x_1}, F_k^{z_1} + P(k, n)^{x_1}\right) = 2,$$

we have

$$2^{(2^{k-1}+1)y-1} \mid F_k^{z_1} - P(k, n)^{x_1}, \quad 2 \mid F_k^{z_1} + P(k, n)^{x_1}, \quad (3.7)$$

or

$$2 \mid F_k^{z_1} + P(k, n)^{x_1}, \quad 2^{(2^{k-1}+1)y-1} \mid F_k^{z_1} - P(k, n)^{x_1}. \quad (3.8)$$

However,

$$2^{(2^{k-1}+1)y-1} > 2^{(2^{k-1}+1)2z_1} > (F_k + F_k - 2)^{z_1} > F_k^{z_1} + P(k, n)^{x_1},$$

a contradiction.

This completes the proof of Theorem 1.

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